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Jauch–Piron Orthomodular Posets and Propositional Systems: A Comparison

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We compare the structures obtained via orthomodular posets and via propositional systems, discussing some examples of the links between them. Despite some analogies, the two structures are fundamentally different.

Quantum logics and propositional systems are different structures built from different points of view. Curiously, in spite of these differences, many examples and some no-go theorems are the same in the two cases. Let us first recall the two approaches before discussing some correspondences between these results.

Quantum logic was initiated by Birkhoff and von Neumann in their famous article, "The logic of quantum mechanics" (Birkhoff and von Neumann, 1936). The primitive notion is the proposition. A proposition for a quantum system was there taken to be a proposition in the ordinary sense, the implication $\Rightarrow$ was noted $<$, and an effort was made to define the "and," "or," and "negation". For them no difficulties seem to appear with the negation; they just defined a convolution $a \mapsto a'$ such that $a'' = a$ and $a < b$ implies that $b' < a'$, and denoted by I the triviality $a \vee a'$ and by O the impossibility $a \wedge a'$. They then assumed that $\mathscr{L}$, the set of propositions, is a lattice and in fact even an orthocomplemented projective geometry. But in the same article they arrive at the conclusion that $a \vee b$ and $a \wedge b$ cannot be directly interpreted as "or" and "and" except for the case where a and b generate a Boolean sublattice as they do in the usual propositional calculus. This is the origin of the notion of a σ-orthomodular poset L. Formally such a structure is a (partially) ordered set L with an

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orthocomplementation $a \mapsto a'$ such that $\bigvee_i a_i$ exists for any countable sequence of pairwise orthogonal elements $a_i \in L$ and the orthomodular law $a < b \Rightarrow b = a \vee (a' \wedge b)$ is satisfied. On such a structure one defines a state as a generalized probability, that is, a map $a \mapsto s(a) \in [0, 1]$ such that for any countable sequence of pairwise orthogonal elements $a_i \in L$ we have that $s(\bigvee_i a_i) = \sum_i s(a_i)$. Using the orthomodular law, it is easy to see that $a < b$ implies that $s(a) \leq s(b)$, a fundamental relation for the interpretation of the state in terms of probabilities.

On the other hand, in the Aerts–Piron approach the primitive notions are experimental project and state. By experimental project (or question in short), Aerts and Piron mean an experiment, as complicated as it may be, that you can eventually perform on the system and where what would be the positive result is defined in advance. Let us consider a collection of such questions (relative to a given well-defined system). Then we can define new question by the following two operations:

The inverse: $\alpha^{\sim}$ is the new question obtained just by interchanging the positive and negative results,

The product: $\prod_i \alpha_i$ is the new question defined for a family of given questions by the following prescription: choose as you wish in the given family one question α_i that you will perform. The positive result is the one defined by the chosen α_i.

It is easy to see that the collection of questions can always be considered to be closed under these two operations. A given question α is called true if the positive result is certain before one decides to perform the experiment. Then $\alpha < \beta$ (α is stronger than β) if β is true each time α is. We write $\mathscr{L}$ for the complete lattice built from the equivalence classes of questions, each equivalence class being called a property. A property is said to be actual when all the equivalent questions of the corresponding class are true.

In this approach the state of the system is defined as the collection of all actual properties. Two states $\mathscr{E}_1$ and $\mathscr{E}_2$ are called orthogonal if there exists a question α for which α is true in $\mathscr{E}_1$ and $\alpha^{\sim}$ is true in $\mathscr{E}_2$. From physical considerations the lattice $\mathscr{L}$ turns out to be not only complete, but also atomistic and orthocomplemented.

As one can see, orthomodular posets (quantum logics) and complete atomistic orthocomplemented lattices are different objects with different structures and different physical interpretations. The first describes what can be said about the system in terms of propositions, in analogy with the usual logics. The second, on the other hand, describes what you can do with the system and in fact an actual property is nothing other than an element of reality in the Einstein sense. No reference is made to classical or quantum physics or to the probability concept.

Now, when one adds more structure these two different mathematical objects become formally very similar.

1. a σ-orthomodular poset is called unital if for every a there is a state s such that $s(a) = 1$. It is a Jauch–Piron logic if every state satisfies the so-called Jauch–Piron condition: If $s(a) = s(b) = 1$, then there exists a $c < a$ such that $c < b$ and $s(c) = 1$. The two following results are well known:

Theorem (Pták and Pulmannová, 1991). A unital Jauch–Piron logic for which every orthogonal set is at most countable is a complete lattice.

Theorem (Rüttimann, 1977). A finite unital Jauch–Piron logic is Boolean.

2. In addition, we can impose that such a complete, atomistic, orthocomplemented lattice is also weakly modular:

$$a < b \Rightarrow a \vee (b \wedge a') = b$$

and satisfies the covering law:

If p is an atom (if p covers 0) and if $a \wedge p = 0$, then $a \vee p$ covers a.

We have then the following two results:

Theorem (Piron, 1964; Amemiya and Araki, 1966). Any complete, orthomodular, atomic lattice satisfying the covering law (that is, any propositional system) is isomorphic to a generalized Hilbert space (or a direct sum of such spaces with possibly non-Desarguian orthogonal planes).

Theorem (Eckmann and Zabey, 1969). Every finite propositional system is Boolean.

Finally, let us recall a result of D. Aerts. The set of properties for the joint system of two separated quantum entities also turns out to be a complete, atomistic, and orthocomplemented lattice, but such a lattice is never weakly modular and never satisfies the covering law. This result causes difficulties in the orthodox quantum logic approach, where weak modularity plays a fundamental role in the interpretation (Moore, 1993).

REFERENCES

Aerts, D. (1982). *Foundations of Physics*, **12**, 1131.

Amemiya, I., and Araki, H. (1966). *Publications Research Institute for Mathematical Sciences A*, **2**, 423.

Birkhoff, G., and von Neumann, J. (1936). *Annals of Mathematics*, **37**, 823.

Eckmann, J.-P., and Zabey, Ph. Ch. (1969). *Helvetica Physica Acta*, **42**, 420.
Moore, D. (1993). Quantum logic requires weak modularity, *Helvetica Physica Acta*, **66**, 471.
Piron, C. (1964). *Helvetica Physica Acta*, **37**, 439.
Piron, C. (1990). *Mécanique quantique: bases et applications*, Presses Polytechniques et Universitaires Romandes, Lausanne, Switzerland.
Pták, P., and Pulmannová, S. (1991). *Orthomodular Structures as Quantum Logics*, Kluwer Academic, Dordrecht.
Rüttimann, G. (1977). *Journal of Mathematical Physics*, **18**, 189.